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Original Article

Flexural Analysis of Isotropic Rectangular Beam Subjected to Uniformly Varying Load Using Seventh Order Shear Deformation Theory for Different Boundary Conditions

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Abstract - *A refined seventh order shear-deformation theory is outlined in current research paper, to investigate bending behavior of deep beam. The displacement field of theory under consideration relies on two variables, where transverse displacement is segmented in bending along with shear. This theory enables direct computation of transverse shear- stresses effectively using constitutive relations, satisfying no shear-stress state on the beam's upper and lower surfaces. Therefore, the shear correction coefficient is not obligatory according to theory. The virtual-work Principle is applied to get the boundary condition and governing expressions. A rectangular isotropic beam under uniformly varying load is recognized for illustration. The analysis is conducted and performed, and the expressions are retrieved for the transverse displacements, normal displacements, normal bending stresses, and transverse shear stresses for various boundary conditions viz simply supported, fixed and cantilever. Results are numerically assessed for a range of length to thickness ratios of beam. Obtained results are represented in the forms of tables and graphs. These results are validated by results of the elementary theory, Timoshenko theory and other higher-order theories available in the literature to substantiate the theory's effectiveness.*

Keywords - *Deep beam, Displacements and stresses, Equation of equilibrium, Seventh order shear deformation theory, Virtual work principle.*

1. Introduction

Shear deformation theory is the fundamental concept in mechanics and structural engineering that helps in the structural analysis of beams and other structural elements. The Euler-Bernoulli [1-4] theory of bending is widely used for this purpose. According to this hypothesis, plane sections that are perpendicular to the neutral axis prior to bending will continue to be normal following the bending condition. It also hinted that there would be no occurrence of transverse shear or transverse-normal strain.

Accordingly, the mentioned theory does not take any notice of the occurrence of transverse shear deformation. This theory quite satisfactorily predicts the behaviour of slender beams throughout the domain of the beam, except at supports. When the beam is short and/or of shear flexible construction, the theory demands more refinement and modification to integrate the influence of transverse-shear; such a modification is called a shear deformation theory.

Galileo and Saint Venant [5] subsequently made substantial contributions to this theoretical framework by tackling problems related to shear and bending forces in beams. Their ground-breaking work offered a thorough solution to complex issues in the field of beam and plate bending, making a significant addition to the classical theory of this field. The foundation for sophisticated models, including rotatory inertia and shear deformation effects in beam analysis, was established by Rankine [6] and Bresse [7].

As noted by Rebello et al. [8], the Timoshenko beam theory [9] in the literature represents a substantial advancement in this knowledge of beam behaviour. Known as the 1st order kinematics-based SD, its kinematics provide more accuracy than the basic beam theory at this point. This theory's comprehensiveness surpasses the constraints of simpler models by offering a more accurate and nuanced picture of beam deformation. Timoshenko was a pioneer in incorporating advanced effects like rotatory inertia and shear-deformation into beam theory.

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The theory is frequently identified as 1st order SD theory or Timoshenko's theory of beam. For the Timoshenko beam theory, Kil'chevskiy [10] and Gol'denveizer [11] were the first to formulate exact boundary conditions. Donnell [12] and Sayir, Mitropoulos [13] investigated techniques depending on the reduction of three-dimensional issues in elastic body mechanics in the creation of complex theories. Dym and Shames [14] studied the Timoshenko theory in detail for thick beams, which are fundamentally two- and three-dimensional elasticity theory problems.

A thorough investigation was carried out by Mindlin and Deresiewicz [15] with the goal of identifying the shear correction coefficient over a variety of beam cross-sections. Cowper was the one who first developed the mathematical expression for this coefficient [16, 17], and then it was enhanced further by Murty [18, 19] by creating a new equation that could be used for various beam profiles.

The shear coefficient's historical evolution has been reported by Kaneko [20], Hutchinson [21], Zillmer and Hutchinson [22], and In addition to providing new information about the shear coefficient in beam bending, Rychter [23, 24] developed a theoretical framework that took rotation into account in connection with mean deflection and relative axial displacement. Building on this framework, Stephen and Levinson [25] presented an improved theory related to the various effects. Renton [26] has presented a theory which broadens its application to the evaluation of stiffness in a variety of beam cross-sections

For beam analysis, advanced shear deformation theories also referred to as higher-order SD—have replaced Elementary Beam Theory (ETB) and 1st order SD. Beyond the scope of traditional beam theory, Soler [27] presented an advanced theory. The accurate planar elasticity equations are solved for the thickness coordinate series. Iyengar and Prabhakara [28] examined beams made of fully elastic, homogenous, and isotropic materials in a different study. The theoretical framework, which was initially created for thick, isotropic, rectangular elastic beams, was extended in scope by Tsai and Soler [29]. By including orthotropic beams, their extension expanded the higher-order theory's range of applications.

According to Essenburg's [30] research, the transverse direction's normal deformation component behaves quadratically along the thickness coordinate. Parallel to this, Leech [31] developed a more sophisticated beam theory by utilizing Hamilton's conceptual framework and the parabolic shear deformation theory. The suggested 3rd-order kinematics theory respects the 0-shear-strain requirements at both edges of the beam. By applying Airy's stress functions to provide elasticity solutions for orthotropic beam bending under polynomial stresses, Silverman [32] made a significant contribution to the subject. These developments were furthered by Levinson [33, 34], who derived an advanced theory for beams that included differential equations for an order four system.

Based on Levinson's kinematic and stress assumptions, Bickford [35] developed an advanced theory for beams that produced a series of 6th-order differential equations. According to Rychter's integration of 2-D linear elasticity theory [36, 37], Levinson's theory is more accurate than Bickford's. Separately, Petrolito [38] included Bickford's theory in a finite element solution designed for thick beams under uniform loading. On the other hand, Rehfield and Murty [39] created a thorough beam theory that took non-classical axial stress effects.

The validity of Rehfield and Murthy's theory was confirmed by Rychter [40], who also emphasized that it applies to anisotropic beams with higher transverse shear deformability. On the other hand, Baluch et al. [41] proposed a theory that considered how transverse-shear and normal strain affect the way beams with isotropic properties bend. Interestingly, the shear correction coefficient is not included in this hypothesis, and the semi-inverse approach is used to refine it. Using a semi-inverse technique, Valisetty [42] developed an improved bending theory intended for beams with solid circular cross-sections.

Separately, Krishna Murty [43] developed a 3rd order beam theory, incorporating concerns for transverse shear strain and nonlinear axial stress. Irretier [44] examined more nuanced effects in the analysis of dynamical theories for linear, identical beams, highlighting the critical role that shear deformation plays. Finite element formulations for isotropic beam bending and vibration were published by Heyliger P.R. and Reddy [45] and Kant and Gupta [46].

Higher-order theories of shear deformation serve as the foundation for these formulations. Using the idea of virtual work, Irschik [47] compared the conventional Bernoulli-Euler theory with more sophisticated beam theories. Furthermore, a link between shear deformable beam and plate solutions and classical solutions was shown by Wang et al. [48]. Dufort, Drapier, and Grediac's paper [49] examines the relationship between various beam theories.

Gao and Wang [50] presented an improved beam theory based on sophisticated plate theory. Ghugal [51-53] developed a beam theory in which transverse shear and normal strain are considered. Using the Papkovich-Neuber solution and the reciprocal theorem, Gao et al. [54] presented a novel approach to solving the beam bending problem. Ghugal and Shimpi [55] created an improved, consistent trigonometric SD theory specifically designed for thick isotropic beam flexure as well as free vibration analysis. Ghugal and Shimpi [56] reviewed a wide range of increased SD theories. Considering the consequences of lateral shear-deformation, Ghugal and

Sharma [57, 58] presented a hyperbolic theory for shear-deformation. Under a thorough investigation of flexural analysis of deep beams, Ghugal and Waghe [59] employed the trigonometric SD theory. Similarly, Shi G. and Voyiadjis [60] presented a unique beam theory with variational consistent boundary conditions.

Sayyad, Ghugal [61] developed a new hyperbolic SD theory for the bending analysis of deep/thick beams. Ghugal and Dahake [62, 63] developed trigonometric SD theory for buckling analysis of thick beams. Ghugal and Gajbhiye [64] presented a V^{th}-order SDT flexure analysis of a simply supported isotropic plate. Mehmet Avcara et al.'s study [65] uses the high-order theory to investigate the natural frequency of FG sandwich beams with various combinations. The validity of the proposed strategy is proven by means of in-depth analyses with data that is available to the public from the open literature.

Fatemeh Sohani and Eipakchi [66] describe an analytical method for figuring out the nonlinear natural frequencies of beams in a different piece of work. Sayyad and Avhad [67] utilized a V^{th}-order theory for curved beams to examine free vibration in functionally graded sandwich beams that were curved in elevation. Avhad and Sayyad [68] performed a static analysis of sandwich beams bent in elevation and laminated composite beams in a similar study. By employing a new quasi-3D polynomial-type beam theory, their research advances comprehension of the structure.

Based on the literature review, the research gap identified as unconstrained sophisticated theories are rarely available for the study of beams that are subjected to shear loads at both their top and bottom. Also, the literature only contains solutions for basic beam bending issues with simple loads and boundary constraints. Further, the practical significance of a number of boundary conditions is overlooked, and the impact of localized stress concentration on the flexural response of shear flexible beams is inadequately addressed.

To address all these research gaps, a seventh-order shear deformation theory is developed. The significance of the present seventh-order theory is that it eliminates the requirement of the shear correction factor, which is necessary for Elementary Theory (ETB) and Timoshenko beam theory (FSDT). The governing equation of the solution is developed from the Principle of virtual work.

The investigation is carried out for the flexural analysis of the fixed beam. Analytical results achieved are evaluated against those documented in the literature. The field has a number of active researchers who focus on shear deformation, resulting in fewer experimental data being available for a few load cases and boundary conditions. Hence, only analytical validation is carried out in the current paper.

2. Theoretical Formulation

2.1. Considered Beam and its Modelling

The beam being studied is a rectangular deep beam with dimensions: length 'L' parallel to the x-axis, a breadth 'b' parallel to the y-axis, and a consistent thickness 'h' parallel to the z-axis. Beam's mid-plane, defined as z = 0, serves as the reference plane. The beam's top surface lies at z = h/2, and its bottom surface is situated at z = -h/2. The beam is considered isotropic, elastic and homogeneous and is subjected to the loading q(x). The figure below illustrates the coordinate system used for the beam's modelling.

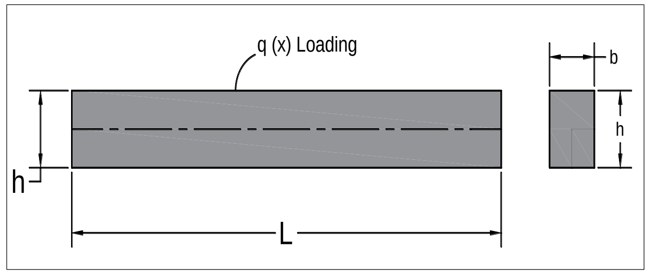

Fig. 1 Beam under consideration

2.2. Assumptions Underlying the Theoretical Formulation

The assumptions underlying the present seventh-order SD theory are as follows.

- Two elements constitute axial displacement: displacement by ETB and the displacement caused by shear deformation.
- Transverse displacement 'w' is acknowledged as a function of longitudinal coordinates in the x-axis direction.
- The axial displacement 'u' is expressed in a way that the axial stress resultant yields merely the bending moment but not the force along the x direction.
- The displacements are minor relative to the thickness of the beam.
- The effect of body forces is disregarded in this analysis.

2.3. Shear Correction Factor

The shear correction factor is an important parameter in the analysis of the beam. It is denoted by "k" and It takes into consideration the uneven distribution of shear stress caused by shear loads over a beam's cross-section. It rectifies the assumption that shear stress is evenly distributed across the beam, which isn't the case with the material and beam sections found in real life.

In view of this, Timoshenko applied the shear correction factor k in the formulation of shear stiffness in his beam theory, which accounts for both bending and shear deformation. For a rectangular beam section, typically, the value of k is considered to be 5/6.

The current Seventh order shear deformation theory accounts for the shear deformation through the sophisticated displacement formulation. The seventh order theory uses complex displacement field that considered the terms for bending as well as shear deformation as mentioned below. The present theory allows non-uniform stress distribution and shear modulus is integrated into the governing differential equation. In view of all these considerations, the use of shear correction factors becomes unnecessary, resulting in a more precise and thorough analysis of beam behaviour subjected to various loads.

2.4. Displacement Field

In light of the above stated assumptions, the refined beam theory's displacement field is expressed in the following manner.

$$u(x,z)=z\frac{dw}{dx}+z\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\phi(x) \quad (1)$$
$$w(x,z)=w(x)$$

Where
'u' - Axial displacement
'w'- Transverse displacement

2.4.1. Normal Strain

$$\varepsilon_x=\frac{du}{dx}=-z\frac{d^2w}{dx^2}+z\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h6}\right)\frac{d\phi}{dx} \quad (2)$$

2.4.2. Shear Strain

$$\gamma_{zx}=\frac{du}{dz}+\frac{dw}{dx}$$

$$\gamma_{zx}=\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-16\frac{z^6}{h^6}\right)\phi(x) \quad (3)$$

Normal axial stress and transverse shear stress are derived through the use of one-dimensional constitutive laws. The mathematical expressions for these stresses are:

$$\sigma_x=E\varepsilon_x$$

$$\sigma_x=-Ez\frac{d^2w}{dx^2}+Ez\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d\phi}{dx} \quad (4)$$

$$\tau_{zx}=G\gamma_{zx}$$

$$\tau_{zx}=G\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-16\frac{z^6}{h^6}\right)\phi(x) \quad (5)$$

2.5. Displacement Field

With the use of the Principle of virtual work and expressions for stresses and strains as above, the governing-equation for the beam under view can be ascertained as below,

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x+\tau_{zx}\delta\gamma_{zx}\right)dx\,dz=\int_{x=0}^{x=L}q(x)\,\delta w\,dx \quad (6)$$

Substituting the expressions for virtual strain and stresses in the above Equation (6) will give the following expression.

$$b\int_{x=0}^{x=L}\int_{z=-\frac{h}{2}}^{z=+\frac{h}{2}}\begin{bmatrix}\left(-Ez\frac{d^2w}{dx^2}+Ez\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d\phi}{dx}\right)\\ \left(-z\frac{d^2\delta w}{dx^2}+z\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d\delta\phi}{dx}\right)\\ +G\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)\phi*\\ \left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)\delta\phi\end{bmatrix}dzdx$$
$$=\int_{x=0}^{x=L}q(x)\delta w dx \quad (7)$$

On further simplification, it becomes

$$b\int_{x=0}^{x=L}\int_{z=-\frac{h}{2}}^{z=+\frac{h}{2}}\begin{bmatrix}Ez^2\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}\\ -Ez^2\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d\phi}{dx}\frac{d^2\delta w}{dx^2}\\ -Ez^2\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d^2w}{dx^2}\frac{d\delta\phi}{dx}\\ -Ez^2\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\frac{d^2w}{dx^2}\frac{d\delta\phi}{dx}\\ +G\phi\delta\phi\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)^2\end{bmatrix}dzdx$$
$$=\int_{x=0}^{x=L}q(x)\delta w dx \quad (8)$$

Integrating the Equation (8) with respect to z gives

$$b\int_{x=0}^{x=L}\begin{bmatrix}EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}-2.67EI\frac{d\phi}{dx}\frac{d^2\delta w}{dx^2}\\ -2.67EI\frac{d^2w}{dx^2}\frac{d\delta\phi}{dx}+7.2EI\frac{d\phi}{dx}\frac{d\delta\phi}{dx}+6GA\phi\delta\phi\end{bmatrix}dx$$
$$=\int_{x=0}^{x=L}q(x)\delta w dx \quad (9)$$

The virtual displacement can be relieved in the region of the beam by integrating the parts Equation [9] so that the fundamental lemma of variational calculus can be obtained

$$\int_{x=0}^{x=L}\left[EI\frac{d^4w}{dx^4}-2.67EI\frac{d^3\phi}{dx^3}-q(x)\right]\delta w dx$$
$$+\int_{x=0}^{x=L}\left[2.67EI\frac{d^3w}{dx^3}-7.2EI\frac{d^2\phi}{dx^2}+6GA\phi\right]\delta\phi dx$$
$$+\left(EI\frac{d^2w}{dx^2}-2.67EI\frac{d\phi}{dx}\right)\frac{d\delta w}{dx}\bigg|_0^L+$$
$$\left(-EI\frac{d^3w}{dx^3}+2.67EI\frac{d^2\phi}{dx^2}\right)\delta w\big|_0^L$$
$$+\left(-2.67EI\frac{d^2w}{dx^2}+7.2\frac{d\phi}{dx}\right)\delta\phi\big|_0^L=0 \tag{10}$$

As the variations δw a δΦ re arbitrary functions over the integration region, the coefficient of their variation can be made zero separately to get the Euler-Lagrange equations (known as equilibrium equations), which will be the governing differential equations of theory in view. Thus, governing differential equations are mentioned below.

$$EI\frac{d^4w}{dx^4}-2.67EI\frac{d^3\phi}{dx^3}-q(x)=0 \tag{11}$$

$$2.67EI\frac{d^3w}{dx^3}-7.2EI\frac{d^2\phi}{dx^2}+6GA\phi=0 \tag{12}$$

At a distance, $x = 0$, $x = L$, the relevant boundary conditions, which is variationally consistent, appear as:

$$\text{Either } V_x=EI\frac{d^3w}{dx^3}-2.67EI\frac{d^2\phi}{dx^2}=0 \text{ or } w \text{ is prescribed} \tag{13}$$

$$\text{Either } M_x=EI\frac{d^2w}{dx^2}-2.67EI\frac{d\phi}{dx}=0 \text{ or } \frac{dw}{dx} \text{ is prescribed} \tag{14}$$

$$\text{Either } M_s=2.67EI\frac{d^2w}{dx^2}+7.2EI\frac{d\phi}{dx}=0 \text{ or } \phi \text{ is prescribed} \tag{15}$$

Where,

V_x - Shear-force resultant

M_x - Bending-moment resultants analogous to ETB

M_s - Moment resultant owing to the influence of transverse SD.

Equations (13) to (15) represent forced or natural boundary conditions, and the right terms indicate kinematic or rigid boundary conditions.

In this manner, governing equations are achieved, and boundary conditions are obtained. The solution of the equations mentioned above describes the beam's static bending behaviour and, at the same time, meets the required boundary conditions.

2.6. Generalized Solution of Governing-Equation

The general solution of w(x) and ϕ(x), which represent transverse displacement and warping function, respectively, is achieved through Equations (11) and (12), which utilize the method for solving linear differential equations with constant coefficients.

By integrating and arranging Equation (11), the following equation is obtained:

$$\frac{d^3w}{dx^3}=2.67\frac{d^2\phi}{dx^2}+\frac{Q(x)}{EI} \tag{16}$$

$Q(x)$ - shear force to beam, represented by

$$Q(x)=\int^x q dx+C_1 .$$

Equation (12) can be articulated in this form:

$$2.67\frac{d^3w}{dx^3}-7.2\frac{d^2\phi}{dx^2}=-\beta\phi \tag{17}$$

Let, $A_0=2.67,\quad B_0=7.2,\quad C_0=6$

Utilizing Equations (16) and (17), a single equation with respect to ϕ can be fetched.

$$\frac{d^2\phi}{dx^2}-\lambda^2\phi=\frac{Q(x)}{\alpha EI} \tag{18}$$

Where the constants α , β and λ in Equations (17) and (18) are,

$$\alpha=\left(\frac{B_0}{A_0}-A_0\right),\quad \beta=\frac{C_0GA}{A_0EI} \text{ and } \lambda^2=\frac{\beta}{\alpha}$$

Equation (18) general solution shall be:

$$\phi(x)=C_2\cosh\lambda x+C_3\sinh\lambda x-\frac{Q(x)}{\beta EI} \tag{19}$$

By entering a value of $\Phi(x)$ into Equation (19). Integrating Equation (19) three times relative to x, transverse displacement's $w(x)$ general solution can be obtained as below

$$EIw(x)=\left(\iiint q\,dx\,dx\,dx\right)+\frac{c_1x^3}{6}+\left(\frac{A_0}{B_0}\lambda^2-\beta\right)\frac{EI}{\lambda^3}\left(c_2\sinh\lambda x-c_3\cosh\lambda x\right)+c_4\frac{x^2}{2}+c_5x+c_6 \tag{20}$$

Where, C_1, C_2, C_3, C_4, C_5, C_6 represents arbitrary constants. These constants are determined by applying the beam's boundary conditions.

3. Illustrative Examples

The theory can be validated by considering the numerical example with loading and different boundary conditions. For static flexure analysis, a homogenous, elastic, isotropic uniform beam of the rectangular cross-section is taken. The beams material properties are ρ= 7800 kg/m^3, E = 210,000 MPa, μ = 0.3

ρ- Density, E – Young's Modulus, μ - Poisson's ratio

3.1. Boundary Condition Analysis

Analysis of various beams under bending problems with uniformly varying loads and different support conditions is presented below.

3.1.1. Simply Supported

A beam with simple supports can freely rotate but cannot translate vertically. The associated support conditions are,

$$\text{At } x=0\,,L \qquad EI\frac{d^2w}{dx^2}=EI\frac{d\phi}{dx}=w=0$$

3.1.2. Fixed Support

A beam with fixed supports has restrictions on both rotation and translation. This configuration generally results in greater stiffness, leading to reduced deflection and increased internal moments. The associated support conditions are as below

$$\text{At } x=0\,,L \qquad w=0,\phi=0,\frac{dw}{dx}=0$$

3.1.3. Cantilever Support

For cantilever beams subjected to uniformly varying loads, the fixed end experiences larger moments and deflections compared to simply supported beams. The associated support conditions are

At Free End

$$EI\frac{d^2w}{dx^2}=EI\frac{d\phi}{dx}=EI\frac{d^3w}{dx^3}=\frac{d^2\phi}{dx^2}=0 \quad \text{at } x=L$$

At Fixed End

$$w=0,\phi=0,\frac{dw}{dx}=0 \quad \text{at } x=0$$

3.2. Example 1: Beam Simply Supported over Span under Linearly Varying Load, q_0 (1-2x/L)

Beam simply supported over span is demonstrated in Figure 2. Experiencing a linearly varying load, $q_0(1-2x/L)$ acting in a downward direction on half of the length from left support and an upward direction on half of the length from right support.

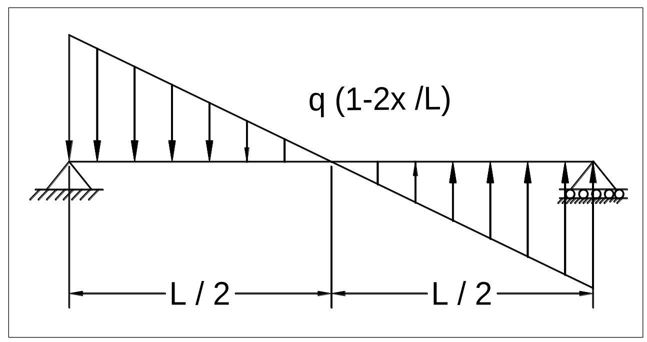

Fig. 2 Beam simply supported over span under linearly varying load, q_0 (1-2x/L)

Being simply supported, the associated support conditions are

$$\text{At } x=0\,,L \qquad EI\frac{d^2w}{dx^2}=EI\frac{d\phi}{dx}=w=0$$

Using the above support conditions, expressions of w (x) and Φ(x), for example, *1*, shall be as follows.

$$w(x)=\frac{q_0L^4}{120EI}\left\{\begin{array}{l}-2\frac{x^5}{L^5}+5\frac{x^4}{L^4}-\frac{10}{3}\frac{x^3}{L^3}+\frac{1}{3}\frac{x}{L}-10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{3}\frac{x^3}{L^3}-\frac{1}{3}\frac{x}{L}\right)+\\ 10\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left[\frac{\cosh\lambda x-\sinh\lambda x-1}{\lambda^2L^2}-\frac{1}{2}\frac{x^2}{L^2}+\frac{1}{\lambda^2L^2}\frac{x}{L}+\frac{1}{2}\frac{x}{L}\right]\end{array}\right\} \tag{21}$$

$$\phi(x)=\frac{q_0L}{GA}\frac{A_0}{C_0}\left(\frac{\sinh\lambda x-\cosh\lambda x}{\lambda L}-\frac{x}{L}+\frac{x^2}{L^2}+\frac{1}{6}\right) \tag{22}$$

Substituting values of w (x), $\Phi(x)$ in the expression of axial stress (σ_x), axial displacement (u), transverse shear-stress (τ_{zx}), final expressions for u, σ_x, τ_{zx} and will be as below:

Axial displacement, u

$$u = \frac{q_0 h}{Eb}\left\{\begin{array}{l} -\frac{z}{h}\frac{L^3}{h^3}\left[\begin{array}{l} -\frac{x^4}{L^4} + 2\frac{x^3}{L^3} - \frac{x^2}{L^2} + \frac{1}{30} + \frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^2}{L^2} - \frac{1}{3}\right) \\ + \frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda L} - \frac{x}{L} + \frac{1}{\lambda^2 L^2} + \frac{1}{2}\right) \end{array}\right] \\ + \frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{z}{h}\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right) \\ \left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda L} - \frac{x}{L} + \frac{x^2}{L^2} + \frac{1}{6}\right) \end{array}\right\} \tag{23}$$

Axial stress, σ_x

$$\sigma_x = \frac{q_0}{b}\left\{\begin{array}{l} -\frac{z}{h}\frac{L^2}{h^2}\left[\begin{array}{l} -4\frac{x^3}{L^3} + 6\frac{x^2}{L^2} - 2\frac{x}{L} + 2\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{x}{L} + \\ \frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\cosh\lambda x - \sinh\lambda x - 1\right) \end{array}\right] \\ + \frac{A_0}{C_0}\frac{E}{G}\frac{z}{h}\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right)\left(\begin{array}{l} \cosh\lambda x \\ -\sinh\lambda x + 2\frac{x}{L} - 1 \end{array}\right) \end{array}\right\} \tag{24}$$

Transverse shear stress τ_{zx}^{CR} obtained from consecutive relationship.

$$\tau_{zx}^{CR} = \frac{q_0}{b}\frac{A_0}{C_0}\frac{L}{h}\left(3 - 4\frac{z^2}{h^2} - 16\frac{z^4}{h^4} - 64\frac{z^6}{h^6}\right) \left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda L} - \frac{x}{L} + \frac{x^2}{L^2} + \frac{1}{6}\right) \tag{25}$$

Transverse shear stress τ_{zx}^{EE} from equilibrium equation:

$$\tau_{zx}^{EE} = \frac{q_0}{b}\left\{\begin{array}{l} -\frac{1}{8}\frac{L}{h}\left(1 - 4\frac{z^2}{h^2}\right)\left[\begin{array}{l} -12\frac{x^2}{L^2} + 12\frac{x}{L} - 2 + 2\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2} + \\ \frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\lambda L\left(\sinh\lambda x - \cosh\lambda x\right) \end{array}\right] \\ -\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\left(\begin{array}{l} \frac{3}{2}\frac{z^2}{h^2} - \frac{1}{3}\frac{z^4}{h^4} - \frac{8}{15}\frac{z^6}{h^6} \\ -\frac{8}{7}\frac{z^8}{h^8} - \frac{1147}{3360} \end{array}\right)\left(\begin{array}{l} \lambda L\sinh\lambda x \\ -\lambda L\cosh\lambda x + 2 \end{array}\right) \end{array}\right\} \tag{26}$$

3.3. Example 2: Fixed Beam under Linearly Varying Load, q_0 (1-2x/L)

A fixed beam is demonstrated in Figure 3. Experiencing a linearly varying load, $q_0(1\text{-}2x/L)$ acting in a downward direction on half of the length from left support and in an upward direction on half of the length from right support.

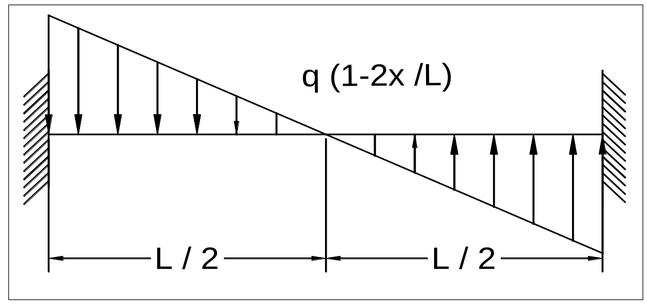

Fig. 3 Fixed beam under linearly varying load, q_0 (1-2x/L)

Being fixed support, the associated support conditions are as below,

$$\text{At } x = 0\text{ , } L \qquad w = 0, \phi = 0, \frac{dw}{dx} = 0$$

Using the above support conditions, the expressions of w (x) and ϕ (x), for example 2, shall be as follows.

$$w(x) = \frac{q_0 L^4}{120EI}\left\{\begin{array}{l} \left(-2\frac{x^5}{L^5} + 5\frac{x^4}{L^4} - 4\frac{x^3}{L^3} + \frac{x^2}{L^2}\right) + \\ 10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{3}\frac{x^3}{L^3} - \frac{1}{2}\frac{x^2}{L^2}\right) - \\ 2\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left[\frac{\sinh\lambda x - \cosh\lambda x + 1}{\lambda L} + \frac{1}{2}\frac{x^2}{L^2} - \frac{x}{L}\right] \end{array}\right\} \tag{27}$$

$$\phi(x) = -\frac{1}{5}\frac{q_0 L}{GA}\frac{A_0}{C_0}\left(\cosh\lambda x - \sinh\lambda x - 1 + 5\frac{x}{L} - 5\frac{x^2}{L^2}\right) \tag{28}$$

Substituting values of w (x), $\Phi(x)$ in the expression of axial stress (σ_x), axial displacement (u), transverse shear-stress (τ_{zx}), final expressions for u, σ_x, τ_{zx} and will be as below:

Axial displacement, u

$$u = \frac{q_0 h}{Eb}\left\{\begin{array}{l} -\frac{1}{10}\frac{z}{h}\frac{L^3}{h^3}\left[\begin{array}{l} -10\frac{x^4}{L^4} + 20\frac{x^3}{L^3} - 12\frac{x^2}{L^2} + 2\frac{x}{L} + 10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(\frac{x^2}{L^2} - \frac{x}{L}\right) \\ -2\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\cosh\lambda x - \sinh\lambda x + \frac{x}{L} - 1\right) \end{array}\right] \\ -\frac{1}{5}\frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{z}{h}\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right)\left(\begin{array}{l} \cosh\lambda x - \sinh\lambda x - 1 \\ +5\frac{x}{L} - 5\frac{x^2}{L^2} \end{array}\right) \end{array}\right\} \tag{29}$$

Axial stress, σ_x

$$\sigma_x = \frac{q_0}{b}\left\{ \begin{gathered} -\frac{1}{10}\frac{z}{h}\frac{L^2}{h^2}\left[\begin{gathered} -40\frac{x^3}{L^3}+60\frac{x^2}{L^2}-24\frac{x}{L}+2+10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(2\frac{x}{L}-1\right)-2\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(\lambda L\sinh\lambda x-\lambda L\cosh\lambda x+1\right) \end{gathered} \right] \\ -\frac{1}{5}\frac{A_0}{C_0}\frac{E}{G}\frac{z}{h}\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\left(\begin{gathered} \lambda L\sinh\lambda x- \\ \lambda L\cosh\lambda x+ \\ 5-10\frac{x}{L} \end{gathered} \right) \end{gathered} \right\} \tag{30}$$

Transverse shear stress τ_{zx}^{CR} obtained from consecutive relationship,

$$\tau_{zx}^{CR} = -\frac{1}{5}\frac{q_0}{b}\frac{A_0}{C_0}\frac{L}{h}\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)\left(\cosh\lambda x-\sinh\lambda x-1+5\frac{x}{L}-5\frac{x^2}{L^2}\right) \tag{31}$$

Transverse shear stress τ_{zx}^{EE} from equilibrium equation:

$$\tau_{zx}^{EE} = \frac{q_0}{b}\left\{ \begin{gathered} -\frac{1}{80}\frac{L}{h}\left(1-4\frac{z^2}{h^2}\right)\left[\begin{gathered} -120\frac{x^2}{L^2}+120\frac{x}{L}-24+ \\ 20\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}-2\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(\lambda^2L^2\cosh\lambda x-\lambda^2L^2\sinh\lambda x\right) \end{gathered} \right] \\ +\frac{1}{5}\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\left(\begin{gathered} \frac{3}{2}\frac{z^2}{h^2}-\frac{1}{3}\frac{z^4}{h^4} \\ -\frac{8}{15}\frac{z^6}{h^6}-\frac{8}{7}\frac{z^8}{h^8} \\ -\frac{1147}{3360} \end{gathered} \right)\left(\begin{gathered} \lambda^2L^2\cosh\lambda x \\ -\lambda^2L^2\sinh\lambda x-10 \end{gathered} \right) \end{gathered} \right\} \tag{32}$$

3.4. Example 3: Cantilever Beam under Linearly Varying Load, q_0 (1-2x/L)

A fixed beam is demonstrated in Figure 4. Experiencing a linearly varying load, $q_0(1-2x/L)$ acting in a downward direction on half of the length from left support and in an upward direction on half of the length from right support.

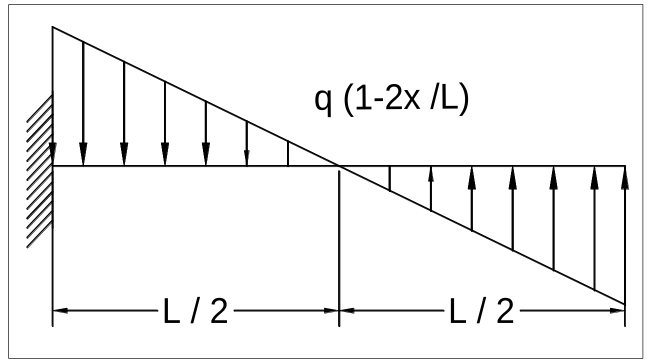

Fig. 4 Cantilever beam under linearly varying load, q_0 (1-2x/L)

Being a cantilever, the associated support conditions are

At Free End,

$$EI\frac{d^2w}{dx^2} = EI\frac{d\phi}{dx} = EI\frac{d^3w}{dx^3} = \frac{d^2\phi}{dx^2} = 0 \text{ at } x = L$$

At Fixed End

$$w = 0, \phi = 0, \frac{dw}{dx} = 0 \text{ at } x = 0$$

Using cantilever support conditions, the expressions for w (x), Φ(x), for example 3, shall be as follows.

$$w(x) = \frac{q_0L^4}{120EI}\left\{\left(-2\frac{x^5}{L^5}+5\frac{x^4}{L^4}-10\frac{x^2}{L^2}\right)+10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{3}\frac{x^3}{L^3}-\frac{x^2}{L^2}\right)\right\} \tag{33}$$

$$\phi(x) = \frac{q_0L}{GA}\frac{A_0}{C_0}\left(\frac{x^2}{L^2}-\frac{x}{L}\right) \tag{34}$$

Substituting values of $w(x)$, $\Phi(x)$ in the expression of axial stress (σ_x), axial displacement (u), transverse shear-stress (τ_{zx}), final expressions for u, σ_x and τ_{zx} will be as below,

Axial displacement, u

$$u = \frac{q_0h}{Eb}\left\{ \begin{gathered} -\frac{z}{h}\frac{L^3}{h^3}\left[-\frac{x^4}{L^4}+2\frac{x^3}{L^3}-2\frac{x}{L}+\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^2}{L^2}-2\frac{x}{L}\right)\right]+ \\ \frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{z}{h}\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\left(\frac{x^2}{L^2}-\frac{x}{L}\right) \end{gathered} \right\} \tag{35}$$

Axial stress, σ_x

$$\sigma_x = \frac{q_0}{b}\left\{\begin{array}{l} -\frac{z}{h}\frac{L^2}{h^2}\left[-4\frac{x^3}{L^3}+6\frac{x^2}{L^2}-2+2\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x}{L}-1\right)\right]+ \\ \frac{A_0}{C_0}\frac{E}{G}\frac{z}{h}\left(3-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}-\frac{64}{7}\frac{z^6}{h^6}\right)\left(2\frac{x}{L}-1\right) \end{array}\right\} \quad (36)$$

Transverse shear stress τ_{zx}^{CR} obtained from consecutive relationship.

$$\tau_{zx}^{CR} = \frac{q_0}{b}\frac{A_0}{C_0}\frac{L}{h}\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)\left(\frac{x^2}{L^2}-\frac{x}{L}\right) \quad (37)$$

Transverse shear stress τ_{zx}^{EE} from equilibrium equation:

$$\tau_{zx}^{EE} = \frac{q_0}{b}\left\{\begin{array}{l} -\frac{1}{8}\frac{L}{h}\left(1-4\frac{z^2}{h^2}\right)\left[-12\frac{x^2}{L^2}+12\frac{x}{L}+2\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\right]- \\ 2\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\left(\frac{3}{2}\frac{z^2}{h^2}-\frac{1}{3}\frac{z^4}{h^4}-\frac{8}{15}\frac{z^6}{h^6}-\frac{8}{7}\frac{z^8}{h^8}-\frac{1147}{3360}\right) \end{array}\right\} \quad (38)$$

4. Results and Discussion

4.1. Results

In the present research article, outcomes associated with axial along with transverse displacement, axial, and transverse stresses are displayed in subsequent unitless numerical form, showing the outcomes in this work. (for aspect ratios 4 and 10). The non-dimensional values can be obtained as,

$$\bar{u} = \frac{Ebu}{q_0 h}, \quad \bar{w} = \frac{10Ebh^3 w}{q_0 L^4}, \quad \bar{\sigma}_x = \frac{b\sigma_x}{q_0}, \quad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}$$

There are two ways for the procurement of transverse shear stresses (τ_{zx}) in the theory. The first one, transverse shear stress, can be found right by adopting constitutive relations, and the second one can be found by integrating a two-dimensional equilibrium equation. The outcomes from both ways are given separately by using different denotations. (τ_{zx}^{CR}) shows shear stress by constitutive relations, and (τ_{zx}^{EE}) shows the same by equilibrium equations. Transverse shear-stress fulfilled the condition of stress less top surface at z = *+h/2* and bottom surface at z = - *h/2* of beam.

An Excel program is developed to obtain the non-dimensional numerical results of displacements and stresses for the expressions from (21) to (38). The values of displacements and stresses are found out at x =0.25L distance from the support against *z/h*. The value of *z/h* varies from -0.5 to +0.5m with the increment of 0.02. The results are obtained for the aspect ratio 4 and aspect ratio 10.

4.1.1. Example 1 Results: Beam Simply Supported over Span under Linearly Varying Load (Refer to Figure 2)

The analysis's outcomes are calculated by assuming section *x-x* is at 0.25*L* from the left end of the beam.

Table 1. Non dimensional - axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ for aspect ratio 'S' 4

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	0.1660	0.1706	1.5944	-0.1113	-0.1225
Ghugal Y M [64]	V order	0.1690	0.1487	1.6064	-0.1160	-0.1205
Krishna Murty [43]	HSDT	0.1651	0.0989	0.9707	-0.1250	-0.1192
Timoshenko [9]	FSDT	0.1709	0.2717	1.5000	-0.0353	-0.1250
Bernoulli-Euler [1, 3, 4]	ETB	0.0488	0.0583	1.5000	--	-0.1250

Table 2. Non dimensional - axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ for aspect ratio 'S' 10

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	0.0537	1.1942	9.4694	-0.2781	-0.3115
Ghugal Y M [64]	V order	0.0537	1.1389	9.4814	-0.2899	-0.3107
Krishna Murty [43]	HSDT	0.0537	1.0154	8.8457	-0.3125	-0.3102
Timoshenko [9]	FSDT	0.0537	4.2448	9.3750	-0.4216	-0.3125
Bernoulli-Euler [1, 3, 4]	ETB	0.0488	0.9115	9.3750	--	-0.3125

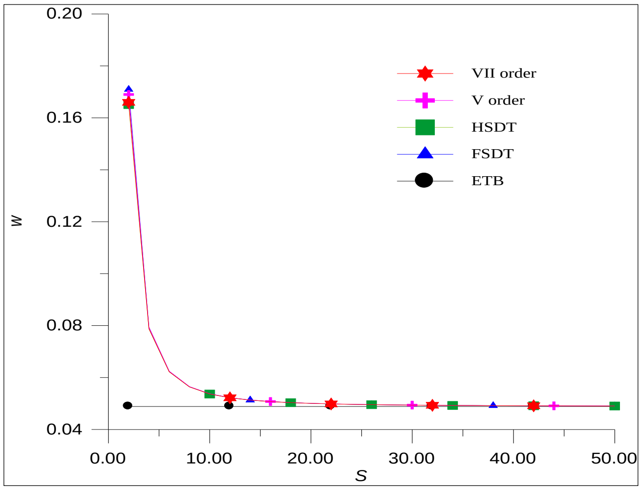

Fig. 5 Transverse displacement ($\bar{w}$) variation through beam thickness (Example 1)

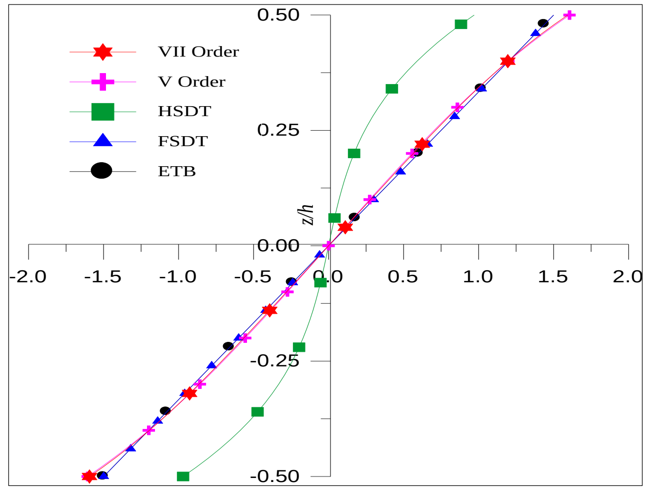

Fig. 8 Aspect ratio 4: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 1)

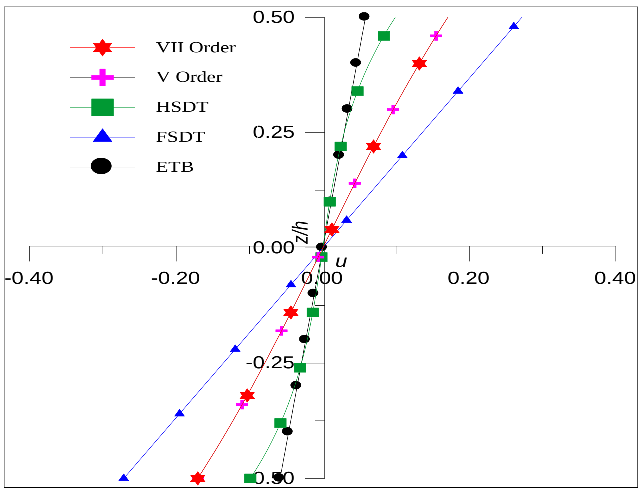

Fig. 6 Aspect ratio 4: Axial displacement ($\bar{u}$) variation through beam thickness (Example 1)

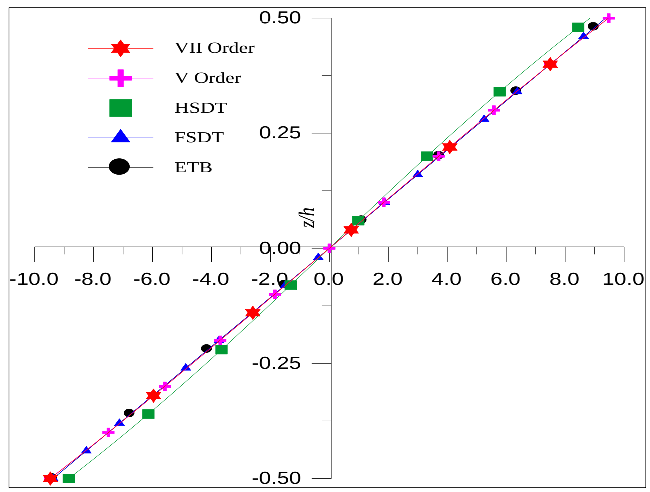

Fig. 9 Aspect ratio 10: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 1)

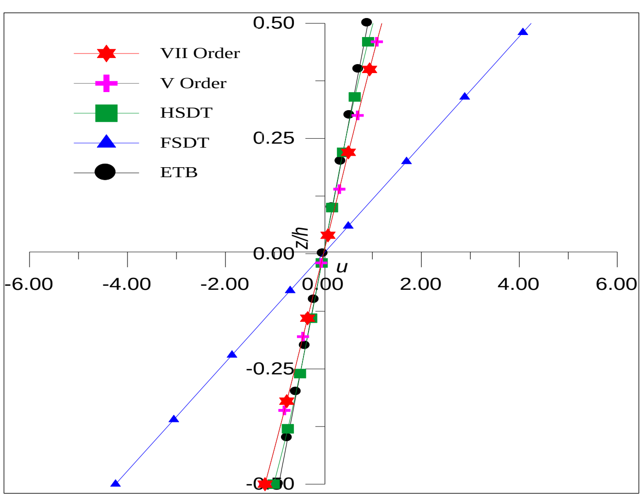

Fig. 7 Aspect ratio 10: Axial displacement ($\bar{u}$) variation through beam thickness (Example 1)

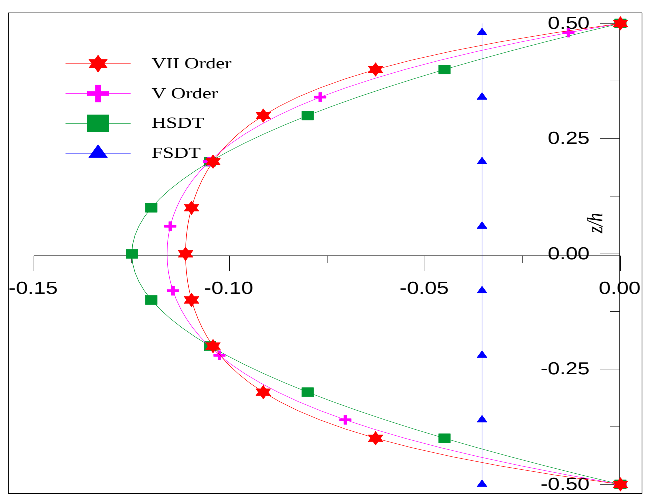

Fig. 10 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 1)

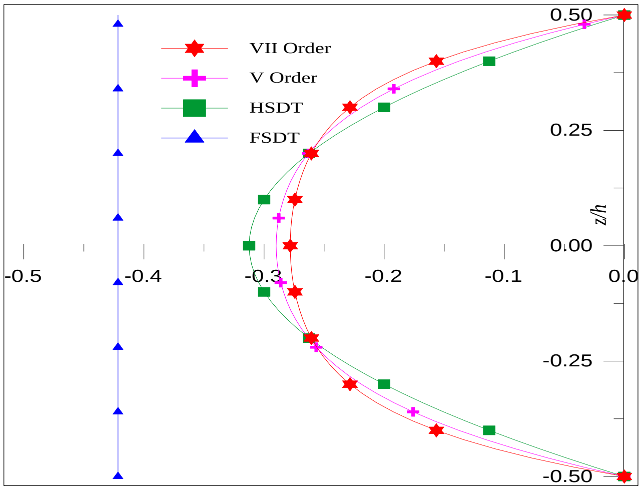

Fig. 11 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 1)

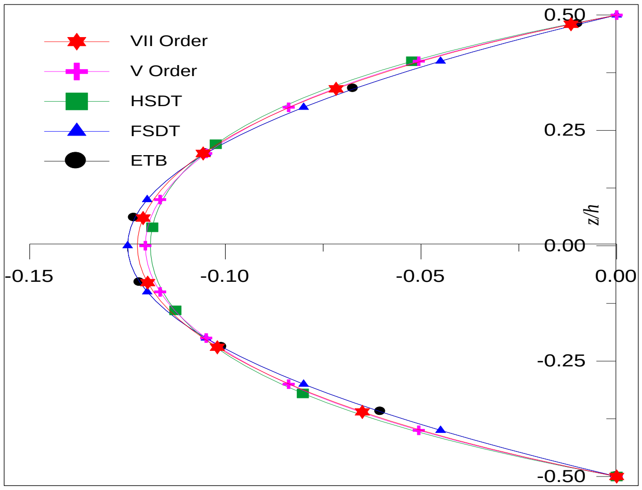

Fig. 12 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 1)

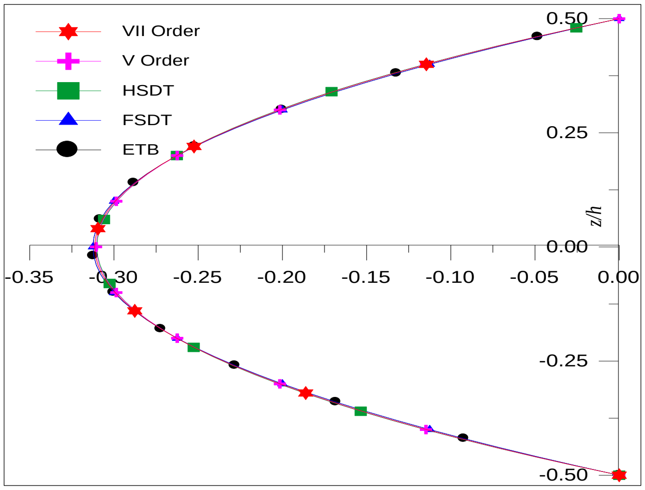

Fig. 13 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 1)

4.1.2. Example 2 Results: Fixed Beam under Linearly Varying Load (Refer to Figure 3)

The analysis's outcomes are calculated by assuming section *x-x* is at 0.25*L* from the left end of the beam.

Table 3. Non dimensional - axial displacement ‘ $\bar{u}$ ’, transverse deflection ‘ $\bar{w}$ ’, axial Stress ‘ $\bar{\sigma}_x$ ’ transverse shear stresses ‘ $\bar{\tau}_{zx}^{CR}$ ’ and $\bar{\tau}_{zx}^{EE}$ for aspect ratio ‘S’ 4

Source	**Model**	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	0.1133	0.2355	1.1168	0.0668	0.0775
Ghugal Y M [64]	V order	0.1193	0.2324	1.1278	0.0696	0.0795
Krishna Murty [43]	HSDT	0.1095	0.2379	1.1513	0.0750	0.0808
Timoshenko [9]	FSDT	0.1860	-0.0750	0.7000	0.0784	0.0750
Bernoulli-Euler [1, 3, 4]	ETB	0.0176	-0.0750	0.7000	--	0.0750

Table 4. Non dimensional - axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ for aspect ratio 'S' 10

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	0.0226	0.3957	4.792	0.1669	0.1885
Ghugal Y M [64]	V order	0.0227	0.4035	4.803	0.1740	0.1893
Krishna Murty [43]	HSDT	0.0226	0.3896	4.826	0.1875	0.1898
Timoshenko [9]	FSDT	0.0094	1.1719	4.375	0.0447	0.1875
Bernoulli-Euler [1, 3, 4]	ETB	0.0176	1.1719	4.375	--	0.1875

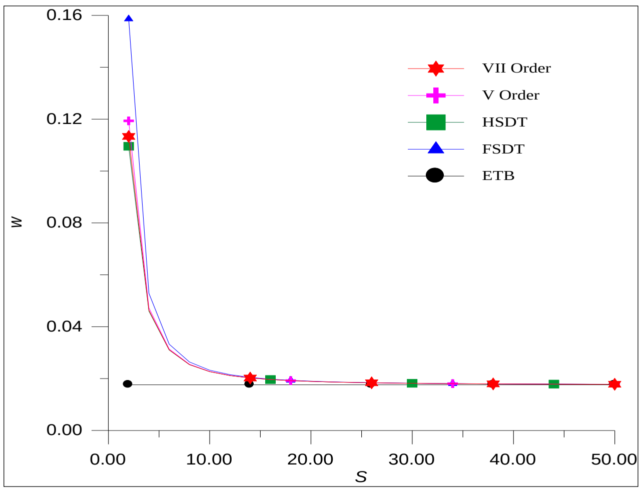

Fig. 14 Transverse displacement ($\bar{w}$) variation through beam thickness (Example 2)

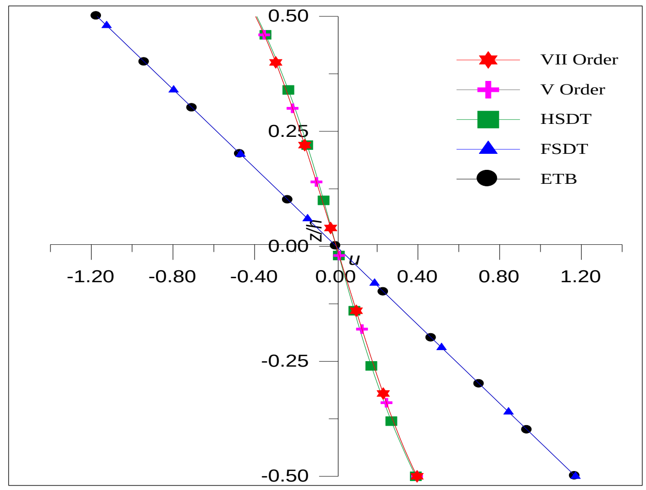

Fig. 16 Aspect ratio 10: Axial displacement ($\bar{u}$) variation through beam thickness (Example 2)

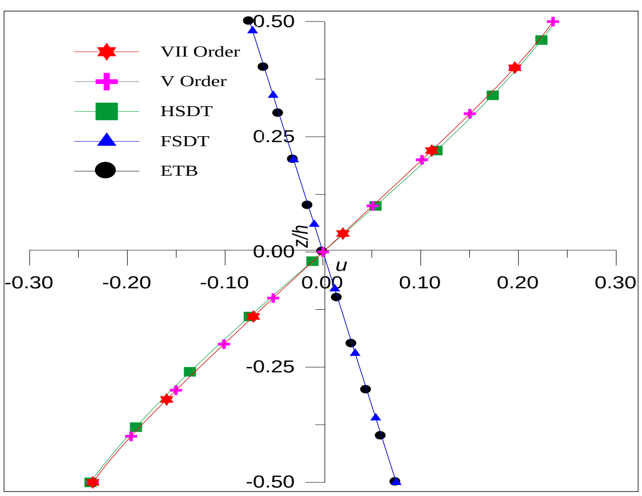

Fig. 15 Aspect ratio 4: Axial displacement ($\bar{u}$) variation through beam thickness (Example 2)

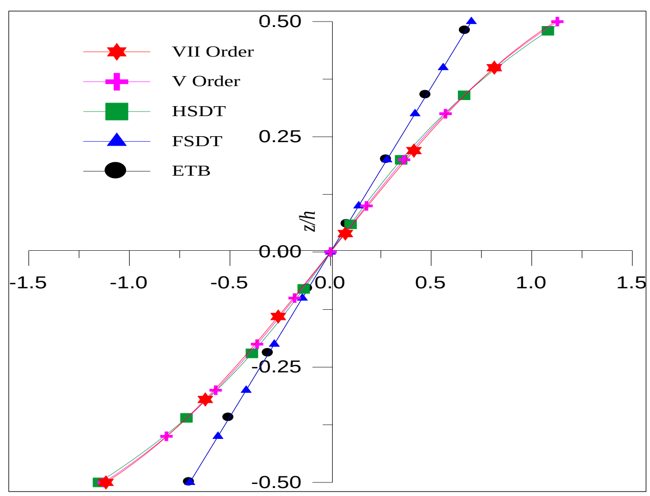

Fig. 17 Aspect ratio 4: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 2)

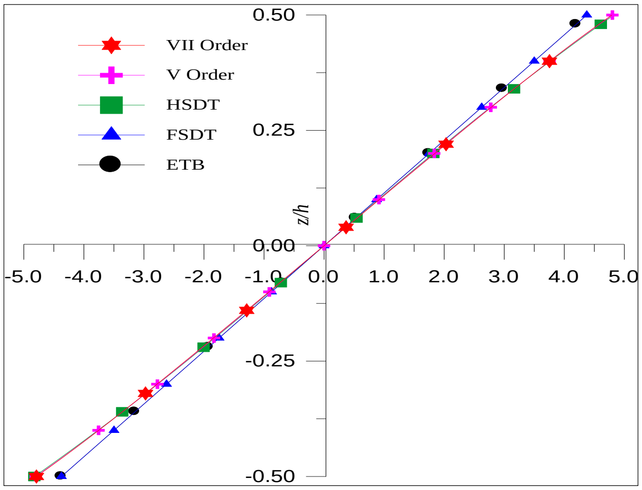

Fig. 18 Aspect ratio 10: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 2)

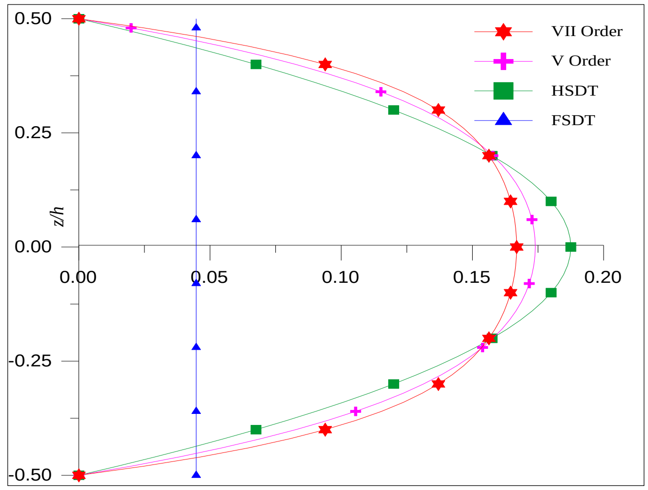

Fig. 20 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 2)

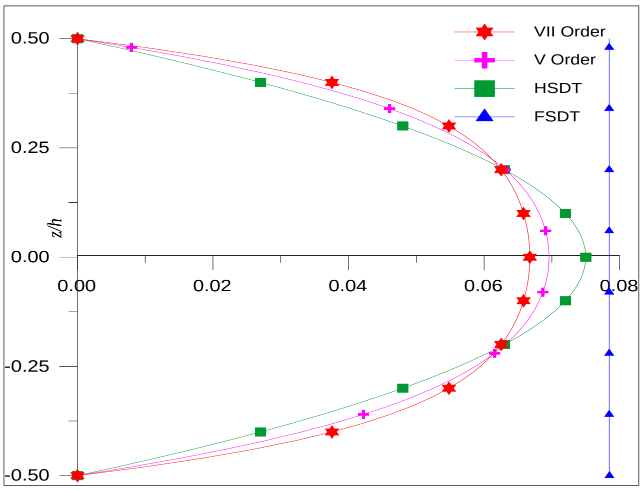

Fig. 19 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 2)

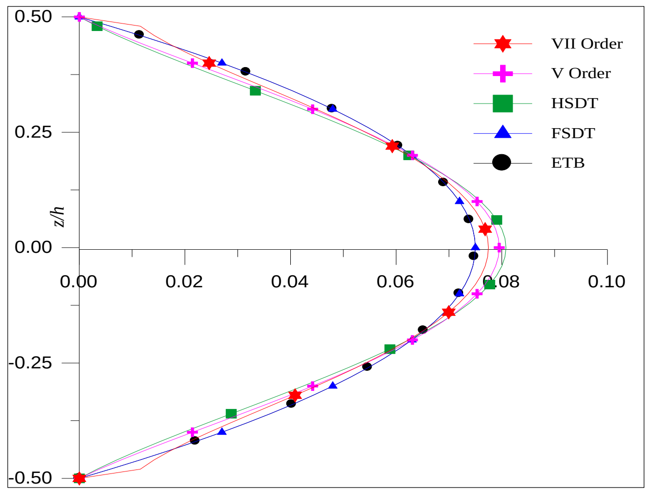

Fig. 21 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 2)

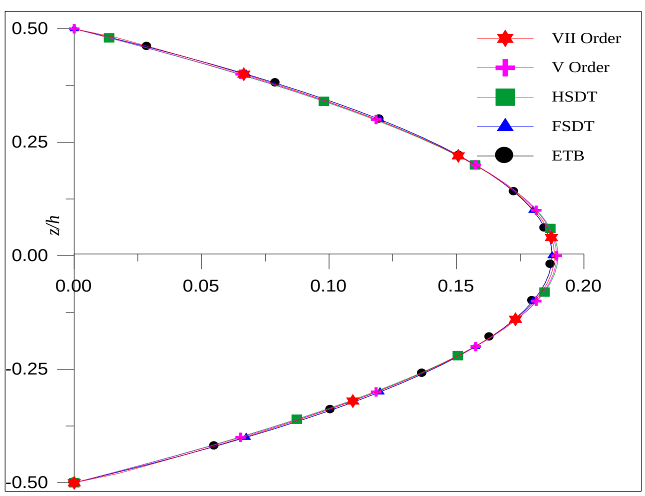

Fig. 22 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 2)

4.1.3. Example 3 Results: Cantilever Beam under Linearly Varying Load (Refer to Figure 4)

The analysis's outcomes are calculated by assuming section *x-x* is at 0.25*L* from the left end of the beam.

Table 5. Non dimensional - axial displacement ‘ $\bar{u}$ ’, transverse deflection ‘ $\bar{w}$ ’, axial Stress ‘ $\bar{\sigma}_x$ ’ transverse shear stresses ‘ $\bar{\tau}_{zx}^{CR}$ ’ and $\bar{\tau}_{zx}^{EE}$ for aspect ratio ‘S’ 4

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	1.0543	16.8468	14.7889	-1.0013	-1.1225
Ghugal Y M [64]	V order	1.0549	16.8634	15.1796	-1.0438	-1.1205
Krishna Murty [43]	HSDT	1.0596	16.9125	15.2179	-1.1250	-1.1192
Timoshenko [9]	FSDT	0.8105	3.1250	13.5000	-2.416	-1.1250
Bernoulli-Euler [1, 3, 4]	ETB	0.6074	15.1250	13.5000	-	-1.1250

Table 6. Non dimensional - axial displacement ‘ $\bar{u}$ ’, transverse deflection ‘ $\bar{w}$ ’, axial Stress ‘ $\bar{\sigma}_x$ ’ transverse shear stresses ‘ $\bar{\tau}_{zx}^{CR}$ ’ and $\bar{\tau}_{zx}^{EE}$ for aspect ratio ‘S’ 10

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	0.6253	240.6325	85.6639	-2.5031	-2.8115
Ghugal Y M [64]	V order	0.6253	240.6741	86.0546	-2.6095	-2.8107
Krishna Murty [43]	HSDT	0.6255	240.7969	86.0929	-2.8125	-2.8102
Timoshenko [9]	FSDT	0.6155	48.8281	84.3750	-3.6583	-2.8125
Bernoulli-Euler [1, 3, 4]	ETB	0.6074	236.3281	84.3750	--	-2.8125

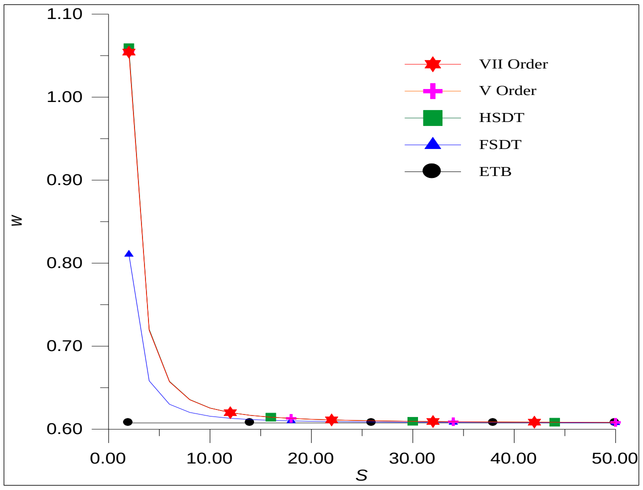

Fig. 23 Transverse displacement ($\bar{w}$) variation through beam thickness (Example 3)

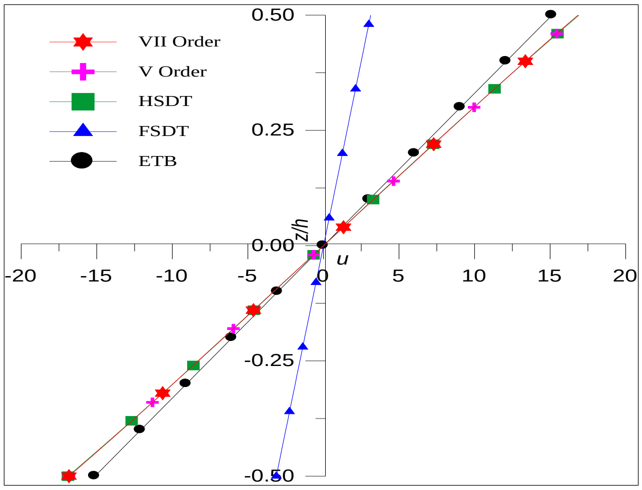

Fig. 24 Aspect ratio 4: Axial displacement ($\bar{u}$) variation through beam thickness (Example 3)

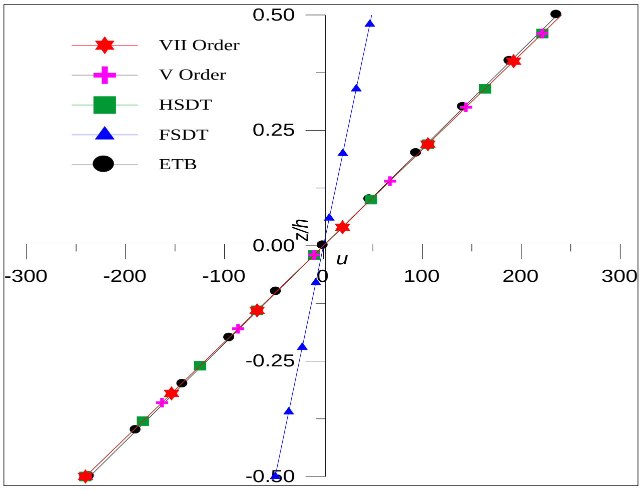

Fig. 25 Aspect ratio 10: Axial displacement ($\bar{u}$) variation through beam thickness (Example 3)

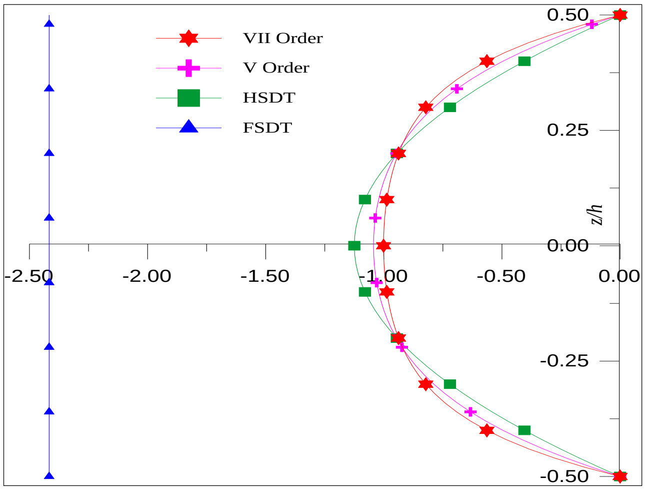

Fig. 28 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 3)

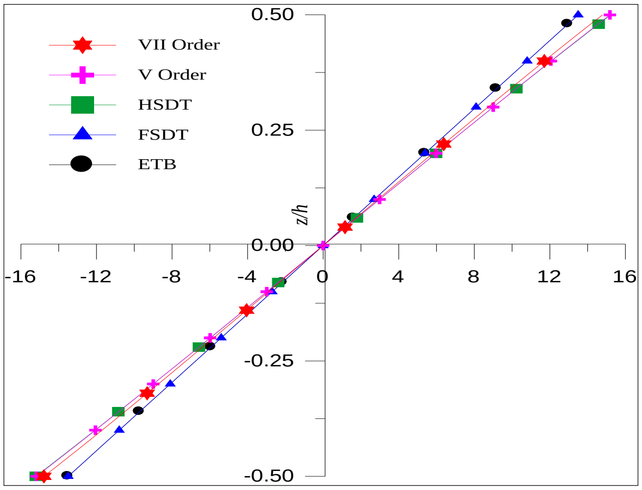

Fig. 26 Aspect ratio 4: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 3)

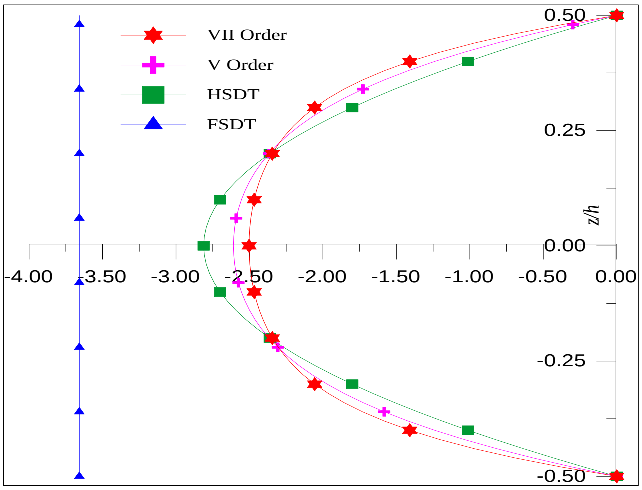

Fig. 29 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) variation through beam thickness adopting constitutive relationship (Example 3)

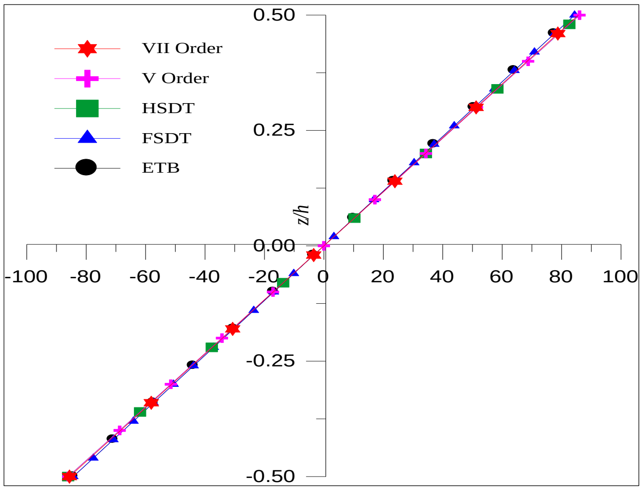

Fig. 27 Aspect ratio 10: Axial Stress ($\bar{\sigma}_x$) variation through beam thickness (Example 3)

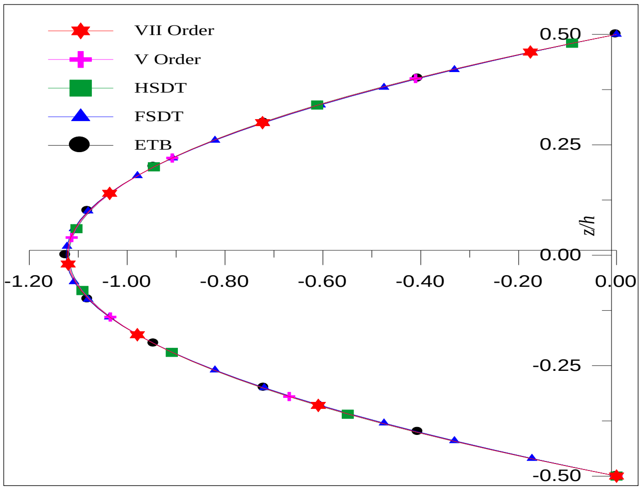

Fig. 30 Aspect ratio 4: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 3)

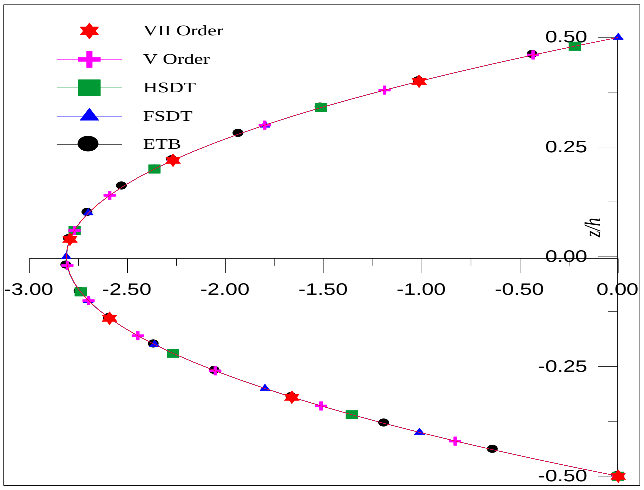

Fig. 31 Aspect ratio 10: Transverse shear stresses ($\bar{\tau}_{zx}^{EE}$) variation through beam thickness adopting equilibrium equation (Example 3)

4.2. Discussion

The results achieved from the current seventh-order theory are evaluated against the results of elementary (Bernoulli Euler) theory (ETB), 1st order theory (Timoshenko theory), higher-order theory (Krishna Murthy) and the order theory of Ghugal.

Figures 1 to 4 show the beam problems considered. Tables 1 to 6 give the non-dimensional maximum values of transverse deflection $\bar{w}$, axial displacement $\bar{u}$, axial stress $\bar{\sigma}_x$, transverse shear stresses $\bar{\tau}_{zx}^{CR}$, and ' $\bar{\tau}_{zx}^{EE}$ ' the simply supported beam, Fixed beam, and cantilever beam under varying load for aspect ratios 4 and 10.

4.2.1. Transverse Displacement $\bar{w}$

Variation in central transverse-displacement $\bar{w}$ having aspect ratio *S* 4 and S10 is shown in Figures 5, 14 and 23 for SS, fixed and cantilever beams with uniformly varying loads. The outcomes from the current seventh order theory are enough similar to other higher order refined theories. However, the deflection anticipated by ETB is lower than that of the present theory, owing to the omission of the effects of shear deformation in ETB. For aspect ratios surpassing 20, the results from all refined theories align with the values predicted by ETB. The graphs for the transverse displacement (Figures 5, 14 and 23) depict that the values $\bar{w}$ reduce as the aspect ratio increases.

4.2.2. Axial Displacement $\bar{u}$

The nature of axial displacement' $\bar{u}$ ' given by current theory, as well as other refined theories, shows the same alignment having aspect ratios 4 and 10. Displacement variation across thickness, according to the current theory, is closely consistent with other refined theories besides the one outlined by Eular theory (ETB) and Timoshenko theory (FSDT). As depicted in Figures 6, 7, 15, 16, 24 and 25 with regard to aspect ratios 4 and 10. Furthermore, in the occurrence of the fixed beam (Figure 15), this displacement component shows a drastic change in its behaviour; this behaviour is found to be reversed with an aspect ratio of 10.

4.2.3. Axial Stress- $\bar{\sigma}_x$

The outcomes given by the current theory for axial -tress ' $\bar{\sigma}_x$ ' in beam thickness and outcomes by other considered refined theories possess exact similarity in their nature of distribution. For the present theory, the nature of variation is nonlinear due to heavy stress concentration, while the variation shown by ETB, along with the First order theory, is linear. ETB, as well as FSDT, failed to find this local stress concentration effect. The behaviour of stresses differs at mid span of beam simply supported over the span with various loads at mid span this is because of stress concentration and this stress distribution matches with other refined theories. From the graphical representation in Figures 8, 9, 17, 18, 26 and 27, it is seen that the nature of the curve of the distribution of axial stress is sharp in lower aspect ratios as compared to higher aspect ratios.

4.2.4. Transverse Shear-Stress $\bar{\tau}_{zx}$

Transverse shear-stress ' $\bar{\tau}_{zx}$ ' is achieved using constitutive relations and integrating equilibrium equations of 2D elasticity. Both methods result in a realistic variation of transverse shear-stress throughout the beam's thickness. Here, the realistic variation signifies the pattern described by refined theories that closely align with the elasticity solution. Here, transverse shear-stress attained by two approaches fulfills the zero shear-stress conditions on the beam's upper and lower surfaces. The variation of transverse shear stress follows a parabolic path along the beam's thickness, which is zero at the top and bottom end and maximum at the centre.

Transverse Shear-Stress from the Consecutive Relationship $\bar{\tau}_{zx}^{CR}$

Transverse shear-stress outcomes calculated from the constitutive relationship in the current theory show a slight mismatch with the alignment of other refined theories; there is no considerable difference between them. In this case, outcomes of FSDT showed straight line which refused to agree with the stress free surfaces when they are graphically represented in the Figures 10, 11, 19, 20, 28 and 29 for aspect ratios 4 and 10 respectively.

Transverse Shear Stress from Equilibrium Equation $\bar{\tau}_{zx}^{EE}$

Transverse shear stress outcomes in current seventh order theory by equilibrium equations and other considered refined theories are excellently follows identical path for transverse shear-stress distribution. Outcomes of ETB and FSDT confirmed no shear stress surfaces, but surprisingly, they are aligned with the refined theories as shown in Figures 12, 13, 21, 22, 30 and 31.

4.2.5. Sensitivity Analysis (Effect of Aspect ratio L/h)

The transverse displacement '$\bar{w}$' remains constant for all the aspect ratios for ETB, while for FSDT, HSDT VI order and VII order, it goes on reducing with increase in aspect ratio and For the aspect ratios greater than 20 all the refined theories converges to the values of classical beam theory (ETB). It is also observed that with slight increase in length to width ratio, there is considerable increase in the non dimensional values of Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ as depicted in the graphs and result tables.

5. Conclusion

The present seventh order theory produces realistic outcomes of displacements, axial stress and shea- stress. Shear stress distribution through the beam thickness is closely aligned with the results drawn from higher-order theories. The impact of stress concentration on transverse shear stress variation is accurately anticipated by this theory using the equilibrium equation of 2D elasticity. In general, from the solved numerical examples, it can be concluded that the present theory maintained a degree of accuracy in the results and is able to forecast the local effect in deep beams having any type of end condition. This validates the theory's effectiveness and reliability.

5.1. Future Work

The present beam theory has good scope for future research. Some of the research areas where this theory can be extended are as follows.

- Dynamic analysis of beams, plates and shells can be carried out.
- This theory can be extended to the analysis of composite beams, plates and shells.
- This theory can be extended to the analysis of laminated beams.
- This theory can be extended to dynamics problems of shear flexible beams.

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